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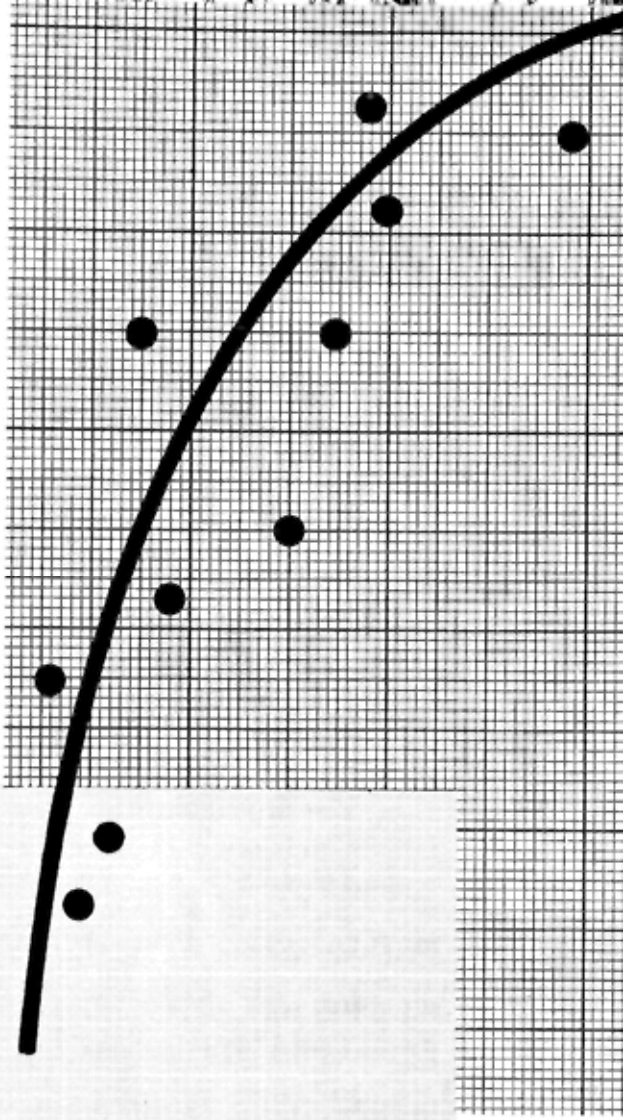

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Simple Methods of Estimating CERTAIN NONLINEAR FUNCTIONS

with emphasis on agricultural data

UNITED STATES DEPARTMENT OF AGRICULTURE Economic Research Service

Washington, D. C.

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Errata Sheet

Agriculture Handbook No. 256: Simple Methods of Estimating Certain Nonlinear Functions with Emphasis on Agricultural Data

Page 2, Equation (2) Substitute α for a

Page 2, Equation (4) Substitute y_{i+1} for y_i+1

Page 4, first paragraph of Part IV, The Gompertz Function; third sentence should read:

"As yield response to nitrogen at low levels of nutrient application frequently manifest this phenomenon, the Gompertz curve is particularly useful. It also may be applicable for estimating trends in technological diffusion, a process that typically exhibits an increasing rate in its early stages and a declining rate in its latter stages (4)."

SIMPLE METHODS OF ESTIMATING CERTAIN NONLINEAR FUNCTIONS WITH EMPHASIS ON AGRICULTURAL DATA

By

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I. INTRODUCTION

This paper presents two elementary methods for fitting three different nonlinear functions to empirical data by means of simple linear regression. Iterative least squares methods which have been developed for estimating parameters of nonlinear functions ~~(2, 8)~~[2] sometimes lead to certain difficulties in application. Because this is the case the much simpler methods developed in this handbook are useful tools for application. The relative merits of this approach versus the nonlinear iterative approach ~~are~~ briefly described in the concluding section.

The Spillman, Gompertz, and Pearl-Reed (logistic) functions ~~are~~ considered. The two methods presented for the Pearl-Reed function are already well known and are given first. ~~Then~~ analogous methods ~~are~~ derived for the Spillman and the closely related Gompertz curves; these apparently have not been presented in the literature.

II. THE PEARL-REED (LOGISTIC) FUNCTION

The logistic curve has been applied by Griliches (4) to the estimation of the rate of adoption of hybrid seed corn, and the author of this handbook has found suitable use for it as a production function.[3] The logistic, like the Gompertz, is an S-shaped or sigmoid curve, and is therefore particularly applicable to the estimation of single input production functions manifesting increasing returns to scale, and to trends in technological diffusion (4, 10). The equation for this function is

$$(1) \qquad y = \frac{\gamma}{1 + \beta e^{-\alpha x}} \qquad \beta,\ \gamma > 0,\ 0 < \alpha < 1$$

in which y is the "dependent" and x the "independent" variable. The parameter γ is the maximum value for y, and it is approached assymptotically as x increases; α determines the rate at which y increases with x, and β determines the value of y when x is zero.

1/ A part of the research on which this report is based was financed by the National Cotton Council.

2/ Underscored figures in parentheses refer to Literature Cited, p. 14.

3/ Day, R. H. Recursive Programing and Production Response. North-Holland Pub. Co., Amsterdam, Contributions to Economic Analysis, 1963.

Method 1: A Transformation of the Dependent Variable

Equation (1) may be placed in the form

$$Z = \log(\gamma / y - 1) = \log \beta - \alpha x \tag{2}$$

If the data are suitable a graphical estimate of γ may be obtained, $\hat{\gamma}$. Letting $Z_i = \log(\hat{\gamma} / y_i - 1)$, $i = 1, \ldots, h$, where h is the number of observations, the regression equation (3) may be fitted by classical least squares.

$$Z = A + Bx \tag{3}$$

Estimates of α and β can now be obtained from the expressions, $\hat{\alpha} = -B$, $\hat{\beta} = e^A$ = antilog A. A similar transformation due to Berkson (1) was used by Griliches.

Note that $\hat{\alpha}$ and $\hat{\beta}$ are not least squares estimates of α and β. Instead, they are derived from the least squares parameters A and B.

Method 2: A Difference Equation for the Dependent Variable

Suppose data are selected in such a way that $x_i = mi$, $i = j, \ldots, j+h-1$. That is, there are h observations available at even intervals of width m for the independent variable. Then equation (1) may be written as

$$y_i = \frac{\gamma}{1 + \beta e^{-\alpha m i}} \tag{4}$$

and

$$y_{i+1} = \frac{\gamma}{1 + \beta e^{-\alpha m (i+1)}}$$

By combining these two expressions the following is obtained,

$$\frac{1}{y_{i+1}} = \frac{1}{\gamma}(1 - e^{-\alpha m}) + e^{-\alpha m} \frac{1}{y_i} \tag{5}$$

Let $Z_i = 1/y_i$, $i = j, \ldots, j+h-1$. Then the difference equation

$$Z_{i+1} = A + BZ_i \tag{6}$$

may be fitted by least squares. From equation (5) it is seen that $A = (1 - e^{-\alpha m}) / \gamma$ and $B = e^{-\alpha m}$. Therefore the estimates of α and γ are $\hat{\alpha} = -(\log B) / m$ and $\hat{\gamma} = (1 - B) / A$

This procedure is due to Rhodes (13) who also suggests a method for estimating β. He utilizes the transformation (2) as follows:

$$\log \beta_i = (\hat{\gamma} / y_i - 1) + \hat{\alpha} m i, \qquad i = j, \ldots, j+h-1. \tag{7}$$

He then takes the mean of the right hand side of this expression as the estimate of $\log \beta$. Thus,

$$\beta = \text{antilog}\ [\Sigma(\hat{\gamma} / y_i - 1) / h + \hat{\alpha} m (h-1)/2]. \tag{8}$$

I have found that this estimator leads to extremely poor fits for some data. Frequently, a much better estimator of β is simply the value which will force the curve through the first (or last or any other) data point (y_j, x_j). Thus

$$\hat{\beta} = e^{\hat{\alpha} mj} [\hat{\gamma}/y_j - 1]. \tag{8}$$

If $x_j = mj = o$, then $\hat{\beta} = (\hat{\gamma}/y_o - 1)$.

A discussion of various methods for estimating the parameters of the logistic may be found in Nair (11).

III. THE SPILLMAN FUNCTION

The Spillman curve has been used by Spillman (14) and Ibach (7) for estimating the yield response of field crops to nitrogen. The latter has developed an ingenious method for graphical estimation of its parameters (6). Paschal and French (12) have presented an iterative least squares method due to Stevens (15) for fitting the function.

The Spillman curve is expressed algebraically as

$$y = \gamma - \beta\alpha^x \qquad 0 < \alpha < 1;\ \gamma, \beta > 0. \tag{9}$$

The parameters α, β, and γ have the same general interpretation as the corresponding parameters for the Pearl-Reed curve. Two methods, analogous to those presented for the logistic, are now derived for estimating these parameters.

Method 1: A Transformation of the Dependent Variable

From equation (9) we obtain

$$Z = \log(\gamma - y) = \log\beta + (\log\alpha)\, x. \tag{10}$$

If data are distributed fortunately a good graphical estimate of γ may be obtained, say $\hat{\gamma}$. Now let (y_i, x_i), $i = 1, \ldots, h$ be a set of h observations of the variables y and x. Let $Z_i = \log(\hat{\gamma} - y_i)$, $i = 1, \ldots, h$. The linear equation

$$Z = A + Bx \tag{11}$$

may now be fitted to the points (Z_i, X_i) $i = 1, \ldots, h$, giving least squares estimates of A and B. From equation (10) it is seen that $\hat{\alpha} = e^B$ = antilog B and $\hat{\beta} = e^A$ = antilog A.

Method 2: A Difference Equation for the Dependent Variable

Let us suppose that the observations (y_i, x_i) are available in such a way that $x_i = mi$, $i = j, \ldots, j + h-1$. That is, the variable x is spaced at even intervals of width m. Then equation (9) may be written

$$y_i = \gamma - \beta\alpha^{mi}, \quad i = j, \ldots, j+h-1, \tag{12}$$

and also

$$y_{i+1} = \gamma - \beta\alpha^{m(i+1)}, \quad i = j, \ldots, j+h-1.$$

By a suitable combination of these two equations the following expression is obtained.

$$y_{i+1} = (1 - \alpha^m)\,\gamma + \alpha^m y_i, \quad i = j, \ldots, j+h. \tag{13}$$

The linear equation

$$y_{i+1} = A + By_i \tag{14}$$

may now be derived from the resulting least squares estimates A and B. Thus $\hat{\alpha} = B^{-m}$ and $\hat{\gamma} = A/(1-B)$.

In a manner analogous to Rhodes' method for estimating β for the logistic the following estimator of β for the Spillman can be obtained.

$$\hat{\beta} = \text{antilog}\,[\Sigma(\hat{\gamma} - y_i)/h - m \log \hat{\alpha}(h-1)/2]. \tag{15}$$

Or by forcing the curve through (y_j, x_j) the first data point, one obtains

$$\hat{\beta} = (\hat{\gamma} - y_j)\,\alpha^{-mj} \tag{15'}$$

$$\text{or } \hat{\beta} = \hat{\gamma} - y_o \text{ if } x_o = o.$$

IV. THE GOMPERTZ FUNCTION

The Gompertz function was found by French (3) to be superior to a great variety of alternative production function forms for the analysis of yield response of corn to nitrogen for a series of Nebraska experiments. The Gompertz is an S-shaped or sigmoid curve, that is, when used as a production function it may show increasing returns to scale. As yield response to nitrogen at low levels of nutrient application frequently manifest this phenomenon, the Gompertz curve ~~also suggests that~~ is particularly useful. It may be applicable ~~to~~ for estimating trends in technological diffusion, a process that typically exhibits an increasing rate in its early stages and a declining rate in its latter stages (4).

The Gompertz function is closely related to the Spillman function as can be observed in its algebraic expression,

$$y = e^{\gamma - \beta\alpha^x} \tag{16}$$

Thus $y' = \log y$ will have the Spillman form if y has the Gompertz form. For this reason the results derived for the former function are directly applicable here.

Method 1: A Transformation of the Dependent Variable

By letting $y'_i = \log y_i$ equation (10) can be obtained from (9). Then $Z_i = \log(\hat{\gamma} - y'_i) = \log(\hat{\gamma} - \log y_i) = \log B + (\log \alpha)x$. The least squares estimates of A and B are obtained from (11). Then the estimates $\hat{\alpha} = e^B$ and $\hat{\beta} = e^A$ are derived just as before. As in the Spillman case an exogenous estimate of γ must be made, probably by a graphical analysis. The only difference is therefore the conversion of the observations of the dependent variable y to logarithms in the case of the Gompertz curve.

Method 2: A Difference Equation for the Dependent Variable

Converting y to logarithms also enables the direct application of method 2 for the Spillman to be used for estimating the parameters of the Gompertz. Thus, if $y'_i = \log y_i$, $i = j, \ldots, j+h-1$ equation (13) may be derived from (16). By fitting (14) the estimates of α and γ are obtained from the least squares parameters A and B. Thus $\hat{\alpha} = B^{-m}$ and $\hat{\gamma} = A/(1-B)$. To estimate β, (15) or (15') may be used.

V. THE FUNCTIONS AS EXPLICIT SOLUTIONS OF DIFFERENCE EQUATIONS

We may note that in the search for a linearizing transformation of the nonlinear functions explicit solutions to several difference equations have been found. These follow from the "even interval data" methods.

The Spillman

The linear, nonhomogeneous difference equation $y_{t+1} = a + b\, y_t$ has as its explicit solution the Spillman function in the form

$$y = \frac{a}{1-b} + \beta\,(b^t) \tag{5.1}$$

in which β is a constant determined by initial conditions on y.

The Gompertz

The nonlinear difference equation $\exp(y_{t+1}) = \exp(a+by_t)$ has as its explicit solution the Gompertz function in the form

$$y = \exp\left[\frac{a}{1-b} + \beta\, b^t\right] \tag{5.2}$$

in which β is determined by initial conditions on y, and in which $\exp(y) = e^y$.

The Logistic

The nonlinear difference equation $y_{t+1}^{-1} = a + by_t^{-1}$ has as its explicit solution the logistic function in the form

$$y = \frac{(1-b)}{a\,(1+\beta e^{t \log b})} \tag{5.3}$$

in which β is determined by initial conditions.

VI. EXAMPLES FOR THE SPILLMAN AND GOMPERTZ FUNCTIONS

The data presented in tables 1 and 2 are used here to illustrate the methods derived above for fitting the Spillman and Gompertz functions. The literature already cited contains examples of the methods described in section II for the Pearl-Reed function. First, three examples are given using the simple production function or relation of one output (oat yield) to one input (nitrogen). Second, two examples are given in which the trend of an aggregate index of mechanical power and machinery is estimated.

The examples demonstrate the utility of these simple methods, but suggest caution in their application. For some types of data, extremely close fits will be obtained; for others, poor results will follow. A graphical presentation of the results will indicate the success or failure of the method. For this reason graphs of both the derived nonlinear curve and the regression on the transformed variables are presented for each example. (Fig. 1 to fig. 10, inclusive.)

Table 1.-Average yields for oats-nitrogen response experiments, Mississippi Delta, 1951-1957

Low rate experiments		High rate experiments	
Nitrogen	Average yield	Nitrogen	Average yield
Pounds	Bushels	Pounds	Bushels
0.0	13.6	0	15.4
7.5	23.4	45	54.4
15.0	34.7	60	60.3
22.5	36.8	75	66.5
30.0	43.5	90	68.0
37.5	49.5	105	66.6
45.0	54.8	120	67.5

Source: Perrin H. Grissom (5).

Table 2.-Index of mechanical power and machinery inputs, 1933-1958

(1947-49 = 100)

Year	Index	Year	Index	Year	Index
--	--	1941	61	1951	127
--	--	1942	66	1952	133
1933	44	1943	69	1953	134
1934	44	1944	70	1954	135
1935	45	1945	74	1955	136
1936	48	1946	80	1956	137
1937	52	1947	89	1957	138
1938	55	1948	100	1958	137
1939	55	1949	111	--	--
1940	58	1950	118	--	--

Source: Ralph A. Loomis and Glen T. Barton (9, table 13).

In the discussion of sections II, III, and IV, natural logarithms or logs to the base "e" were used. However, Naperian logarithms, logs to the base 10, will serve just as well. The latter were in fact used for all computations contained in this section.

Example 1: The high rate data from table 1 are used to illustrate method 1 for the Spillman function. The estimated curve and the original data are presented in figure 1. These data indicate a strong nonlinear relationship, and a good " eyeball " estimate of the upper asymptote, γ, apparently can be made. γ was estimated by eye to be 68.1. Figure 2 shows the simple regression of the transformed yield variable $z_j = \log (68.1 - y_j)$ on nitrogen level. The transformation is strongly linear except for the z value at 90 pounds of nitrogen. This "flyer" is due to the fact that the logs of values between zero and 1 are negative, and decrease very rapidly to $-\infty$ as z gets close to zero. As, $z_5 = 68.1 - 68 = 0.1$, this strong negativity results.

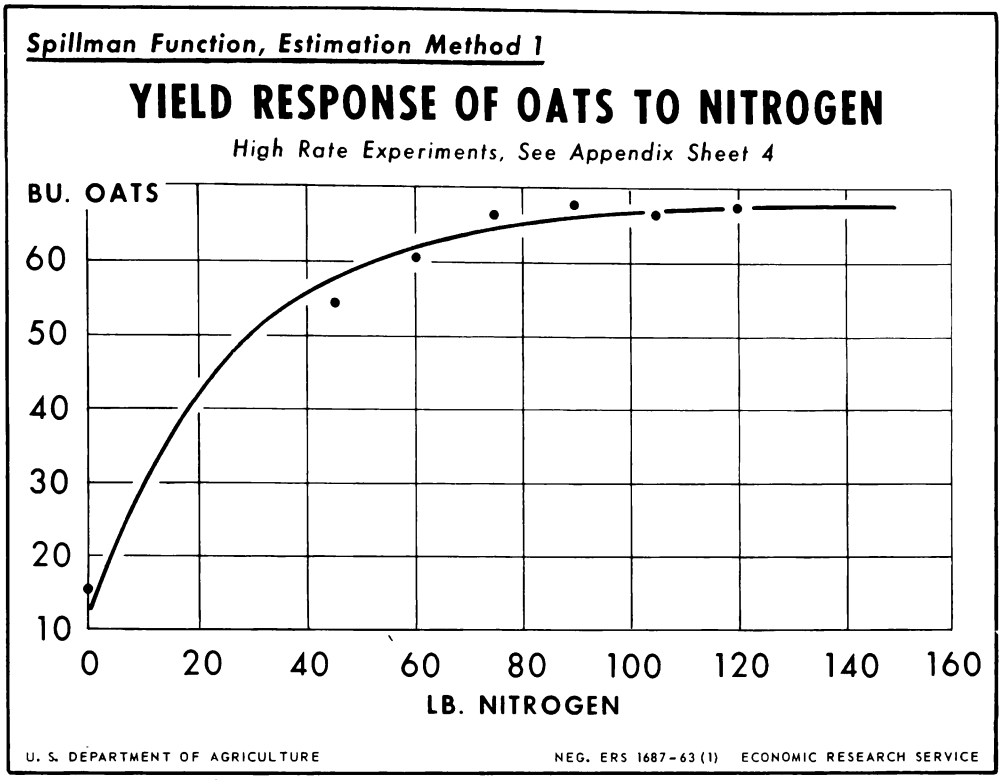

Figure 1

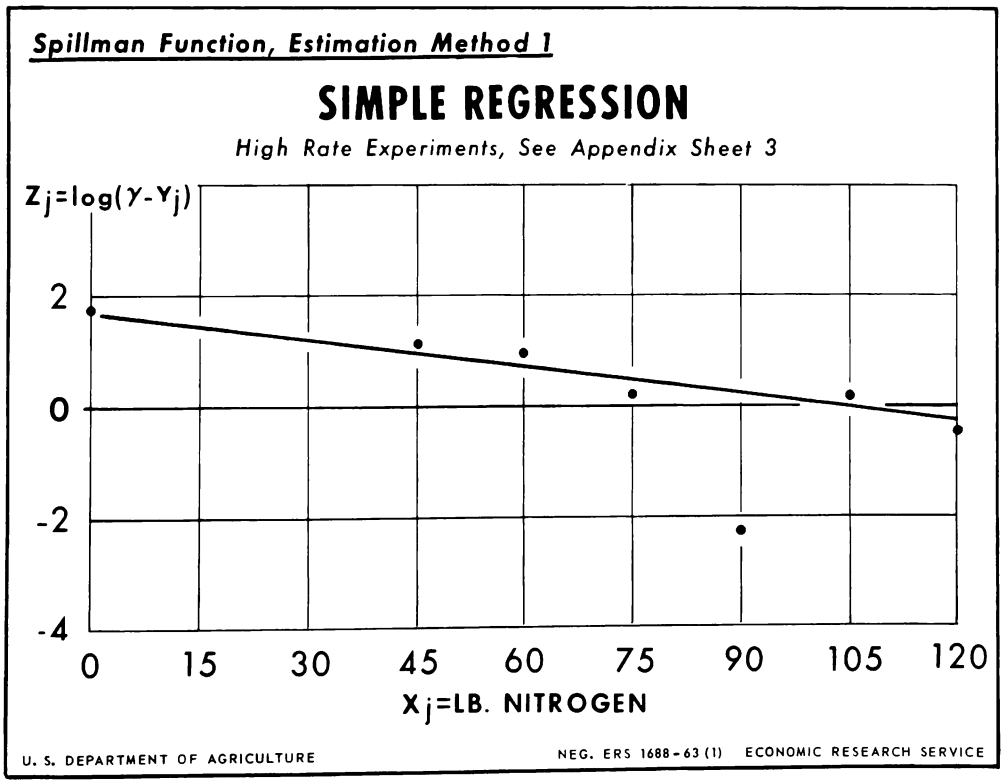

Figure 2

Such a flyer demonstrates the extreme care one should take in applying this particular transformation. In this example and those to follow all such flyers were omitted in the computation of the regression coefficients A and B.

Example 2: Figure 3 shows the result of applying method 2 for the Spillman function for the low rate data of table 1. Unlike the high rate data, these 7 observations are distributed at even nitrogen intervals of 7.5 pounds and method 2 is applicable. Figure 4 shows the simple regression of yield on "lagged" yield $y_{i+1} = A+B\,y_i$. This relation is also strongly linear. The flyer problem is not present for these data and all seven observations were used. The strong linear relationship shown in figure 4 is equivalent to serial correlation in a time series variable. The serial correlation of yields spaced at equal nitrogen levels is due to a causal relation between yield levels and nitrogen applications. It is exploited here to estimate the causal or structural relation represented by the nonlinear Spillman function.

Method 2 yields estimates of γ and α. The β coefficient was estimated by using equation 15' with j = 0, that is, $\beta = \hat{\gamma} - y_0$

Example 3: To illustrate the use of the Gompertz function using method 1, the low and high rate experiments of table 1 were combined. The data and the estimated Gompertz production function are shown as figure 5. Substituting 10 for e in the various Gompertz equation logarithms to the base 10 were used. Thus the upper asymptote was estimated as $68 = 10^{\gamma}$ so that $\hat{\gamma}$ was chosen as 1.83251. Figure 6 illustrates the simple regression of the transformed variables, $z_j = \log(\hat{\gamma} - \log y_j) = A + Bx_j$ where x_j was the nitrogen level. A close linear relation is evident in this transformation but the use of logs, as in example 1, introduces a flyer in the transformed data. At 90 pounds of nitrogen the experimental yield is 68.0, the same as the eyeball estimate of $10^{\hat{\gamma}}$. Consequently, the antilogarithm is $-\infty$. This data point was omitted. The occurrence of this flyer suggests, as in example 1, that the transformation for method 1 is linearizing for a considerable range of the data, but not for points close to the maximum attainable value. A superior estimate might be obtained by choosing only data points in the 0-60 pound nitrogen level range. Examination of the graph of the transformed variable is again seen to be an important precautionary supplement to a straightforward application of method 1.

Example 4: The data from table 2 were used to illustrate method 2 for the Gompertz function. Figure 8 indicates a strong linear serial correlation of the $z_j = \log y_j$ variables, that is, the logs of the aggregate power and machinery index. However, the derived nonlinear relationship shown in figure 7 is extremely poor. While this might not have been guessed from a graphical analysis of the transformed data (fig. 8), the reason manifests itself in the original data. The latter do form a sigmoid or S-shaped curve, and the Gompertz is itself a function of this kind. However, the Gompertz best represents data which increase rapidly at first and then taper off into a long and more gradual approach to the upper asymptote. Yield response data typically follow such a pattern. The index considered, however, has an opposite pattern, that is, a gradual increase climaxed by an abrupt approach to an asymptote.

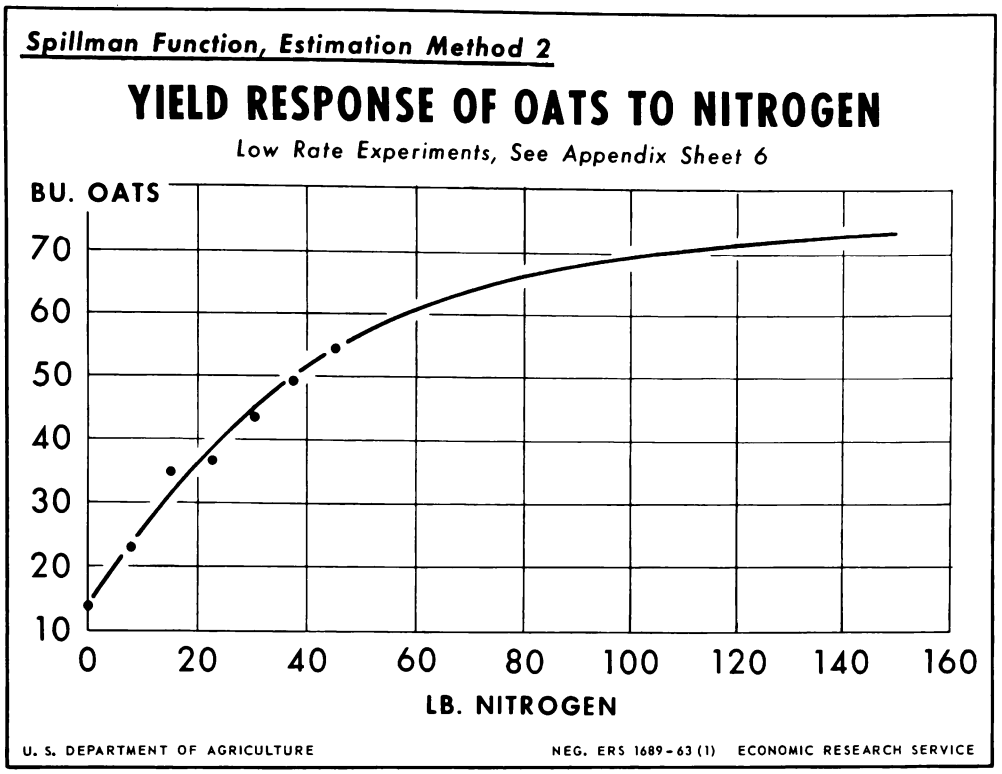

Figure 3

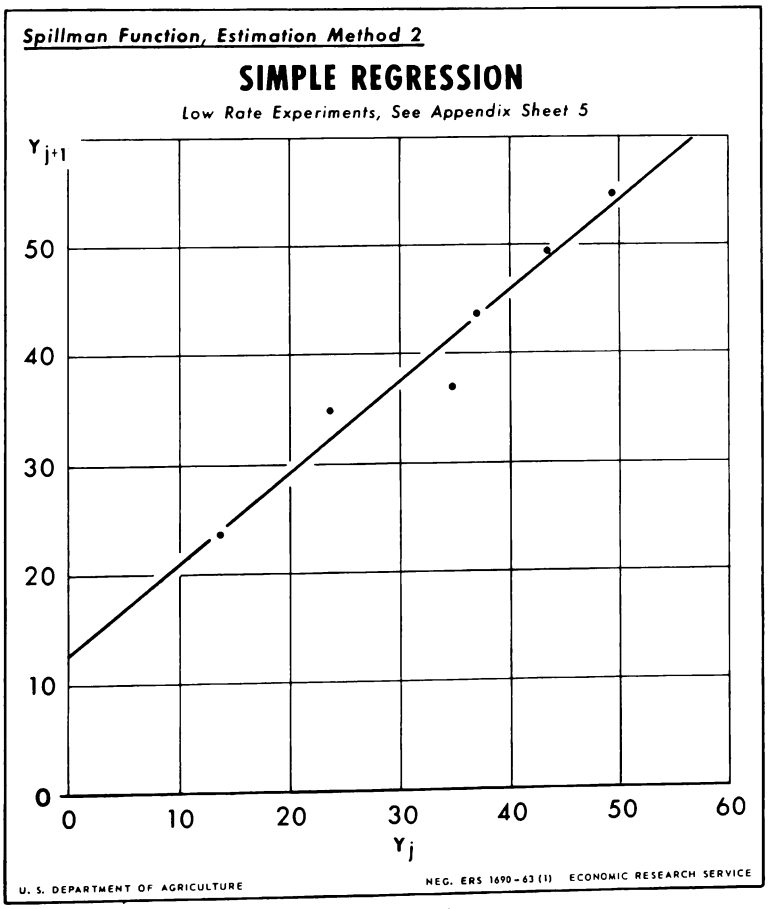

Figure 4

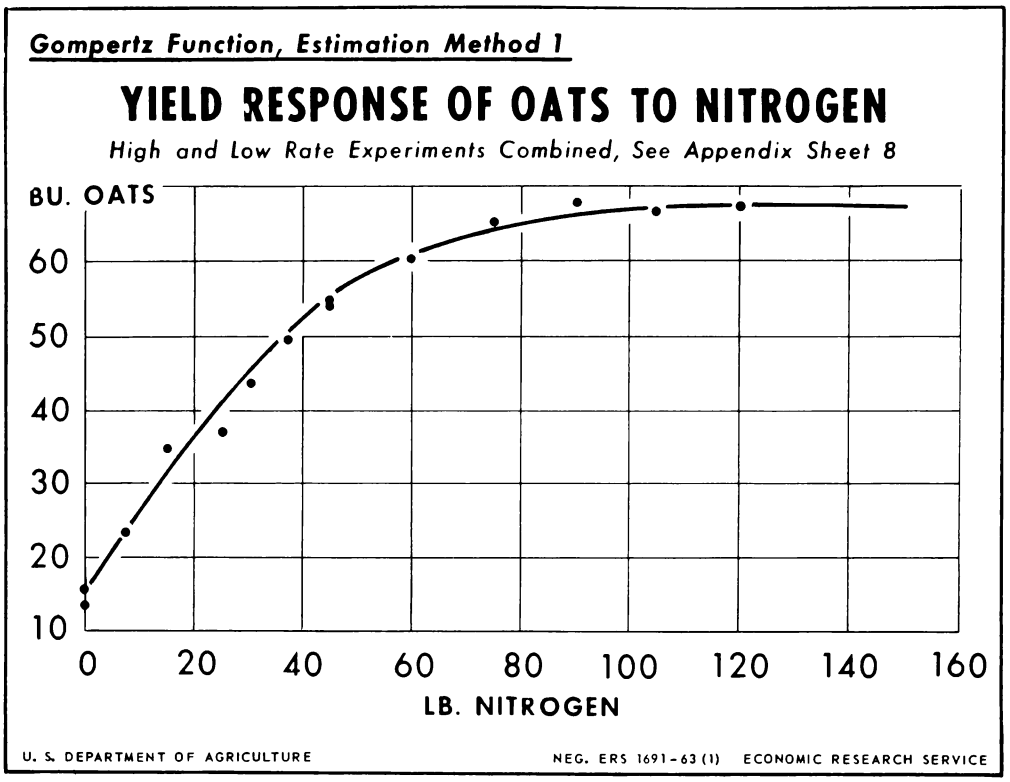

Figure 5

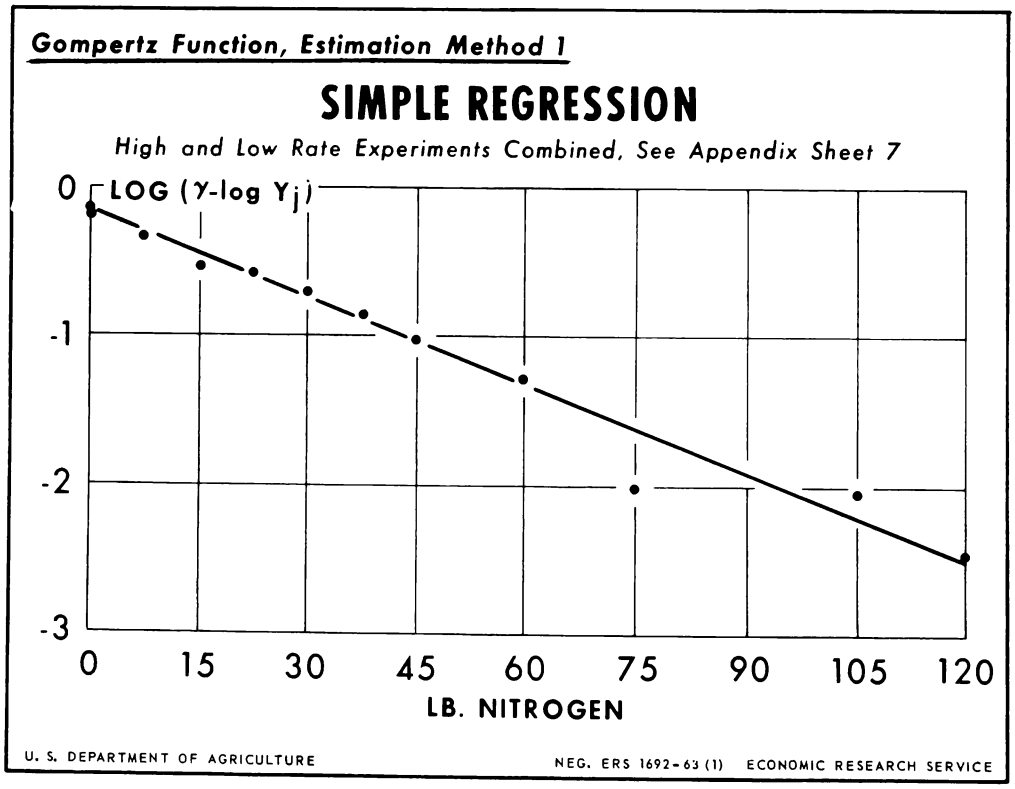

Figure 6

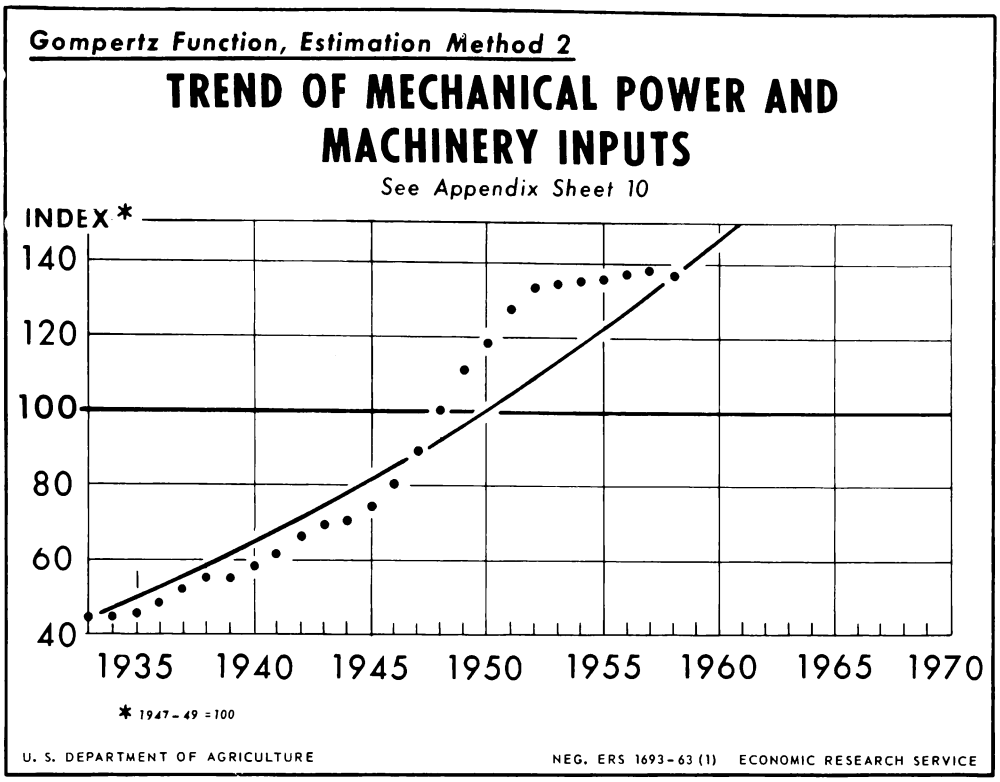

Figure 7

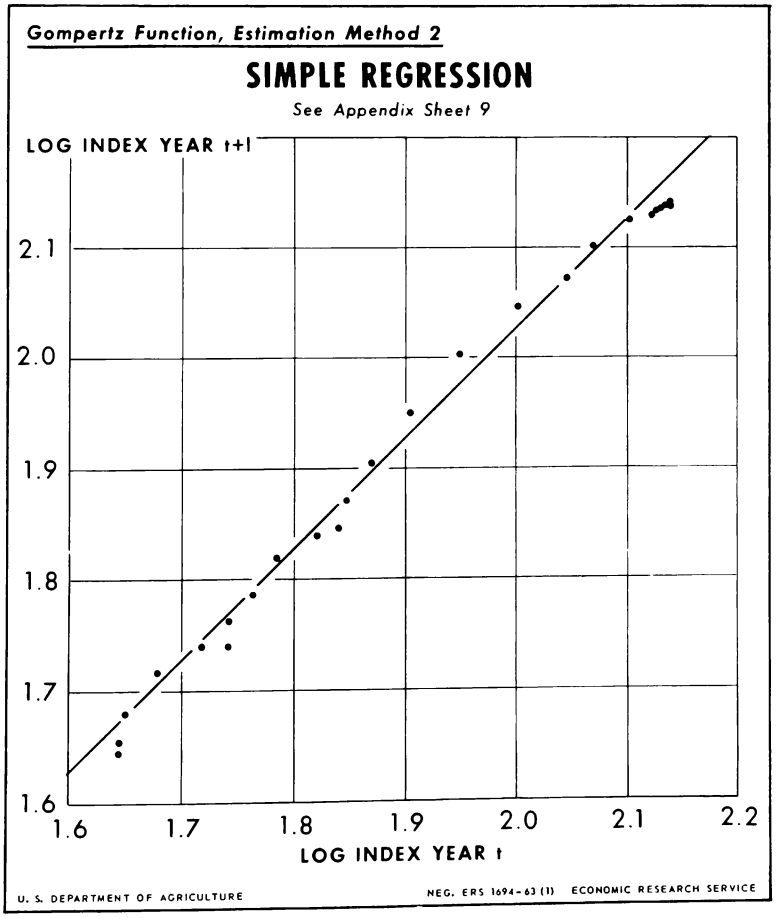

Figure 8

The logistic curve, which is a symmetric sigmoid function, would better represent the data, but even it would not be adequate to describe the asymmetry of this particular example. Again we are reminded to supplement the application of the techniques of this paper by graphical analysis of the data.

Example 5: The Spillman function is not sigmoid but could be used to represent the upper portion of an S-shaped curve. As the Gompertz and Logistic curves are not suited for representing the index of power and machinery over the period 1933-1958, the Spillman could be used to represent the upper portion of the trends. Figure 9 shows the result using method 2. As in example 1, the strong serial correlation of the variable enables the parameters of the Spillman to be estimated with some accuracy. Then a close fit to the index for the sub-period is obtained.

VII. THE RELATIVE MERITS OF LINEAR TRANSFORMATION VS. ITERATIVE METHODS

The fitting of nonlinear functions to empirical data has been greatly facilitated in recent years by the advent of computer programs 4/ for iterative least squares estimation techniques (2, 8). These techniques generally begin with an initial guess for the parameters of the particular nonlinear function to be fitted. They then choose increments by which to modify the initial guess in such a way as to diminish the sums of squares of deviations of the data from the function fitted. One such modification is called an iteration. For well behaved functions and well behaved data a series of such iterations will converge to the least squares estimates.

Such techniques, however, have been found through extensive practical use to have four disadvantages.

(1) Certain types of data lead to convergence difficulties. Briefly, it has been found that if data are not sufficient to approximate an extensive portion of the function, numerous local optima in a least squares sense may be found by beginning with various initial guesses for the parameters. 5/ One is thus faced with the problem of finding all such least squares estimates and picking the best one.

(2) The computer programs involved usually require a specially programed subroutine for the particular function being used. Not all research workers who desire to use nonlinear functions will be interested in, or have the time to program such subroutines.

(3) The iterative procedures may consume a substantial amount of computer time relative to the linear transformation method.

4/ The following references describe several such programs: Booth, G. W., and Peterson, T. I. Non-linear Estimation (Princeton – IBM), in Mathematics and Applications Department, Internatl. Bus. Mach. Corp., Forecasting by Generalized Regression Methods (WL NLI), White Plains, N. Y., Feb. 1959; Gardner, R. S. A Least Squares Iteration. U. S. Naval Ord. Test Sta., IBM SHARE Distrib. No. 934, Prog. No. 145-20, China Lake, Calif., June 1960; and Wood, P. B. Nonlinear Least Squares – (OR NLLS). IBM Appl. Sci. Spec. Prog. Group, SHARE Distrib. No. 837, Union Carbide Nuclear Co., Oak Ridge, Tenn.

5/ See Gardner, footnote 4, A Least Squares Iteration.

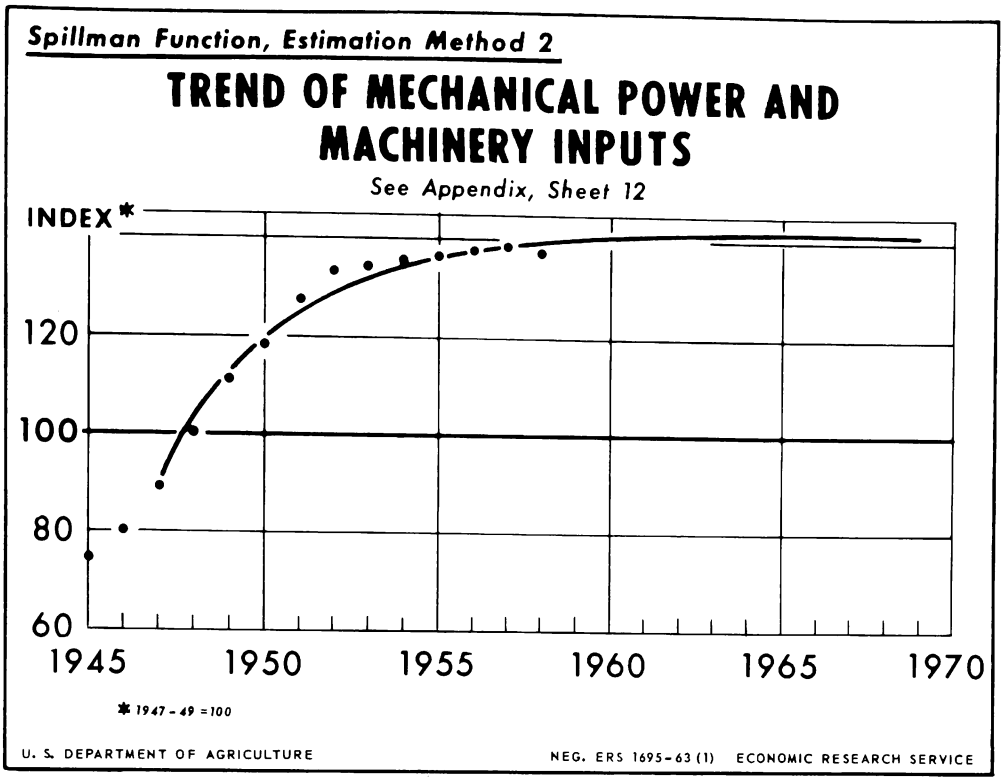

Figure 9

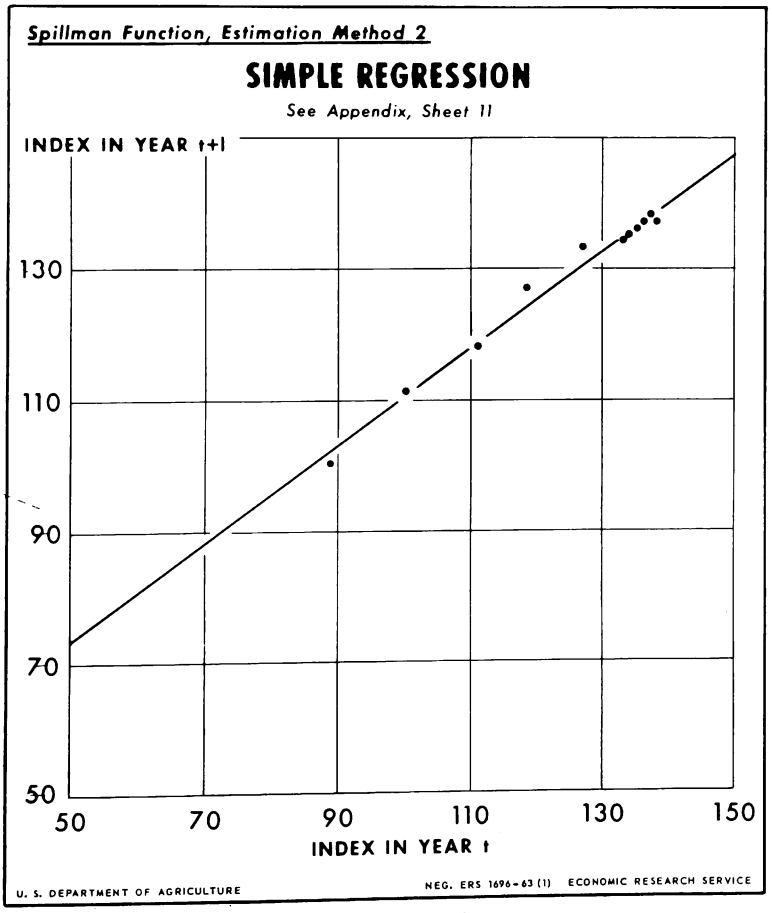

Figure 10

(4) Finally, iterative techniques are not well adapted for use on desk calculators, a disadvantage that may be serious to research workers who do not have an adequate scientific computer at their disposal.

The chief advantage of iterative techniques is their convergence if the data are suitable to least squares parameter estimates when the error or shock term is additive.

The disadvantages of transformation methods are (1) that they do not give least squares estimates for additive error terms, and (2) that some of the parameters may have to be exogenously estimated either "by eye" or some other means, perhaps one statistically arbitrary. But their ease of application, particularly to data which fall short of the range of values needed to approximate a sufficient range of the function, may frequently justify their use. Finally, one may observe that the methods presented in this handbook may be used to find initial parameter values which may then be improved by the iterative methods to which we have referred.

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APPENDIX

The computations involved in the methods described in this paper require only a desk calculator and a table of natural or common logarithms. Computation forms were prepared to facilitate the various steps in estimating the coefficients. These forms are shown in this appendix. Forms for fitting the logistic function are included, though empirical examples were not prepared. The data and all computations are then provided on forms appropriate for each example presented in section VI.

Flyers described in the text, which were omitted in the computation of the parameters are indicated by asterisks.

COMPUTATION SHEET
for
Estimation of Logistic Parameters
(Method 1)

Description of Data: Sample computation sheet for fitting the Pearl-Reed or logistic curve by means of Method 1.

$\hat{\gamma}$ = _____

(1)	(2)	(3)	(4)	(5)	(6)
X_j variable	Y_j variable	$\frac{\hat{\gamma}}{Y_j}$ = _____ (2)	(3) - 1.0	Log (Col. 4) = Z_j	(5) x (1)

ΣZ_j = _____ ΣX_j = _____ $\Sigma X_j Z_j$ = _____

ΣX_j^2 = _____ $(\Sigma X_j)^2$ = _____ $\Sigma X_j \ \Sigma Z_j$ = _____

$$B = \frac{\Sigma X_j Z_j - \frac{1}{n} \Sigma X_j \ \Sigma Z_j}{\Sigma X_j^2 - \frac{1}{n} (\Sigma X_j)^2} = \text{\_\_\_\_\_}$$

$$A = \frac{1}{n} [\Sigma Z_j - B \Sigma X_j] = \text{\_\_\_\_\_}$$

$\hat{\alpha}$ = _____ $-B$ = _____ $\hat{\beta}$ = Antilog A = _____

COMPUTATION SHEET
for
Estimation of Logistic Parameters
(Method 2)

Description of Data: Sample computation sheet for fitting Pearl-Reed or logistic curve by means of Method 2.

(1)	(2)	(3)	(4)
Y_j	$Z_j = \frac{C*}{Y_j}$	$Z_{j+1} = \frac{C}{Y_{j+1}}$	Col. 2 • Col. 3 $Z_j \cdot Z_{j+1}$

C = (scale factor)*

$\sum_{j=1}^{n-1} Z_j =$ _____

$\sum_{j=1}^{n-1} (Z_j^2) =$ _____

$(\sum_{j=1}^{n-1} Z_j)^2 =$ _____

$\sum_{j=1}^{n-1} Z_j Z_{j+1} =$ _____

$\sum_{j=1}^{n-1} Z_j \sum_{j=1}^{n-1} Z_{j+1} =$ _____

$$B = \frac{\sum_{j=1}^{n-1} Z_j Z_{j+1} - \frac{1}{(n-1)} \sum_{j=1}^{n-1} Z_j \sum_{j=1}^{n-1} Z_{j+1}}{\sum_{j=1}^{n-1} Z_j^2 - \frac{1}{(n-1)} (\sum_{j=1}^{n-1} Z_j)^2} = \text{\_\_\_\_\_}$$

$$A = \frac{1}{c(n-1)} \left[\sum_{j=1}^{n-1} Z_{j+1} - B \sum_{j=1}^{n-1} Z_j \right] = \text{\_\_\_\_\_}$$

$$\hat{\alpha} = \frac{-\log B}{M} = \text{\_\_\_\_\_} = \text{\_\_\_\_\_} \qquad \hat{\gamma} = \frac{(1-B)}{A} = \text{\_\_\_\_\_}$$

*Scale factor introduced for computational convenience. Note appearance of "c" in equation for the constant, A.

TABLE FOR COMPUTING VALUES OF LOGISTIC FUNCTION

Description of Data: Sample computation sheet for computing values of logistic curve. Parameters from sheets 1a and 1b.

α = _____

β = _____

γ = _____

(1)	(2)	(3)	(4)	(5)	(6)
X variable	$-\alpha \cdot X$	Antilog Col. (2)	$\beta \cdot$ Col. 3	1.0 - Col. 4	$\gamma \div$ Col. 5

COMPUTATION SHEET
for
Estimating Spillman Parameters
(Method 1)

Description of Data: Computations for Example 1, Figure 2. Columns 2 and 3 taken from Table 1, high rates. $\hat{\gamma}$ = 68.1

(1)	(2)	(3)	(4)	(5)	(6)
observation number j	X_j variable	Y_j variable	γ - Col. 3	Log Col. 4 = Z_j	Col. 5 x Col. 2
1	0	15.4	52.7	1.72181	0
2	45	54.4	13.7	1.13672	51.2
3	60	60.3	7.8	.89209	53.5
4	75	66.5	1.6	.20412	15.3
*5	90	68.0	.1	-2.303	-207.3
6	105	66.6	1.5	.17609	18.5
*7	120	67.5	.6	-.511	-61.3

ΣZ_j = 4.13083 ΣX_j = 285.0 $\Sigma X_j Z_j$ = 138.5

ΣX_j^2 = 22,275.0 $(\Sigma X_j)^2$ = 81,225.00 $\Sigma X_j \Sigma Z_j$ = 1177.28655

$$B = \frac{\Sigma X_j Z_j - \frac{1}{n} \Sigma X_j \Sigma Z_j}{\Sigma X_j^2 - \frac{1}{n} (\Sigma X_j)^2} = -.0161$$

$$A = \frac{1}{n} [\Sigma Z_j - B \Sigma X_j] = 1.7438$$

$\hat{\alpha}$ = Antilog B = .9840 $\hat{\beta}$ = Antilog A = 55.44

*Flyers as described in text omitted from computations.

TABLE FOR COMPUTING VALUES OF SPILLMAN FUNCTION

Description of Data:

Computation for Example 1 Figure 1.
Parameters from Sheet 3.

α = .9840
β = 55.44
γ = 68.1
Log α = -.0161

(1)	(2)	(3)	(4)	(5)
X variable	Log α · Col. 1	Antilog Col. 2	β · Col. 3	γ - Col. 4
0	0	1.0000	55.4400	12.7
7.5	-.1208	.7572	41.9792	26.1
15.0	-.2415	.5634	31.2349	36.9
22.5	-.3622	.4274	23.6951	44.4
30.0	-.4830	.3241	17.9681	50.1
37.5	-.6038	.2458	13.6272	54.5
45.0	-.7245	.1864	10.3340	57.8
60.0	-.9660	.1073	5.9487	62.2
75.0	-1.2075	.0617	3.4206	64.7
90.0	-1.4490	.0355	1.9681	66.1
105.0	-1.6905	.0200	1.1088	67.0
120.0	-1.9320	.0117	.6486	67.4
150.0	-2.4150	.0038	.2107	67.9

COMPUTATION SHEET
for
Estimating Spillman Function
(Method 2)

Description of Data: Computations for Example 2, Figure 4. Columns 1 and 2 from Table 1, low rates. (M=7.5)

(1)	(2)	(3)
Y_j variable	$Y_j + 1$ variable	Col. 1 × Col. 2
13.6	23.4	318.24
23.4	34.7	811.98
34.7	36.8	1276.96
36.8	43.5	1600.80
43.5	49.5	2153.25
49.5	54.8	2712.60

$\sum_{j=1}^{n-1} Y_j = \underline{201.5}$

$\sum_{j=1}^{n-1} Y_j^2 = \underline{7633.35}$

$(\sum_{j=1}^{n-1} Y_j)^2 = \underline{40,602.25}$

$\sum_{j=1}^{n-1} Y_j \; Y_j+1 = \underline{8873.83}$

$\sum_{j=1}^{n-1} Y_j \sum_{j=1}^{n-1} Y_{j+1} = \underline{48,904.05}$

$\sum_{j=1}^{n-1} Y_{j+1} = \underline{242.7}$

$$B = \frac{\sum_{j=1}^{n-1} Y_j \; Y_{j+1} - \frac{1}{(n-1)} \sum_{j=1}^{n-1} Y_j \sum_{j=1}^{n-1} Y_{j+1}}{\sum_{j=1}^{n-1} Y_j^2 - \frac{1}{(n-1)} (\sum_{j=1}^{n-1} Y_j)^2} = \underline{.8342}$$

$$A = \frac{1}{(n-1)} \left[\sum_{j=1}^{n-1} Y_{j+1} - B \sum_{j=1}^{n-1} Y_j \right] = \underline{12.4358}$$

$$\hat{\alpha} = \text{Antilog} \left(\frac{\log B}{M} \right) = \underline{.9761} \qquad \hat{\gamma} = A/(1-B) = \underline{75.0048}$$

TABLE FOR COMPUTING VALUES OF SPILLMAN FUNCTION

Description of Data:

Computations for Example 2 Figure 3.
Parameters from sheet 5.

α = .9761
β = 61.4
γ = 75.0
Log α = -.0242

(1)	(2)	(3)	(4)	(5)
X variable	Log α · Col. 1	Antilog Col. 2	β · Col. 3	γ - Col. 4
0	0	1.0000	61.4000	13.6
7.5	-.1815	.8340	51.2076	23.8
15.0	-.3630	.6956	42.7098	32.3
22.5	-.5445	.5801	35.6181	39.4
30.0	-.7260	.4838	29.7053	45.3
37.5	-.8954	.4084	25.0758	49.9
45.0	-1.0890	.3366	20.6672	54.3
60.0	-1.4520	.2341	14.3737	60.6
75.0	-1.8150	.1628	9.9959	65.0
90.0	-2.1780	.1133	6.9566	68.0
105.0	-2.5410	.0788	4.8383	70.2
120.0	-2.9040	.0548	3.3647	71.6
150.0	-3.6300	.0265	1.6271	73.4

COMPUTATION SHEET
for
Estimating Gompertz Function
(Method 1)

Description of Data: Computations for Example 3, Figure 6. Column 2 taken from Table 1. $\hat{\gamma}$ = 1.83251 = Log 68.0

(1)	(2)	(3)	(4)	(5)	(6)	(7)
j =	X_j variable	Y_j variable	Log Col. 3	γ- Col. 4	Log Col. 5 = z_j	Col. 6 x Col. 2
1	0	13.6	1.13354	.69897	-.15552	0
2	0	15.4	1.18752	.64499	-.19044	0
3	7.5	23.4	1.36922	.46329	-.33442	-2.5082
4	15.0	34.7	1.54033	.29218	-.53462	-8.0193
5	22.5	36.8	1.56585	.26666	-.57349	-12.9035
6	30.0	43.5	1.63849	.19402	-.71220	-21.3660
7	37.5	49.5	1.69461	.13790	-.86012	-32.2545
8	45.0	54.8	1.73878	.09373	-1.02812	-46.2654
9	45.0	54.4	1.73560	.09691	-1.01362	-45.6129
10	60.0	60.3	1.78032	.05219	-1.28233	-76.9398
11	75.0	66.5	1.82282	.00969	-2.01368	-151.0260
*12	90.0	68.0	1.83251	0	$-\infty$	$-\infty$
13	105.0	66.6	1.82347	.00904	-2.04383	-214.6022
14	120.0	67.5	1.82930	.00321	-2.49349	-299.2188

ΣZ_j = -13.2359 $\quad \Sigma X_j$ = 562.5 $\quad \Sigma Z_j X_j$ = -910.7165

ΣX_j^2 = 41,793.75 $\quad (\Sigma X_j)^2$ = 316,406.25 $\quad \Sigma X_j \Sigma Z_j$ = 7445.1938

$$B = \frac{\Sigma Z_j X_j - \frac{1}{n} \Sigma Z_j \Sigma X_j}{\Sigma X_j^2 - \frac{1}{n} (\Sigma X_j)^2} = -.0194$$

$$A = \frac{1}{n} [\Sigma Z_j - B \Sigma X_j] = -.1787$$

$\hat{\alpha}$ = Antilog B = .9564 $\quad \hat{\beta}$ = Antilog A = .6627

*See note on Sheet 3.

TABLE FOR COMPUTING VALUES OF GOMPERTZ FUNCTION

Description of Data:

Computations for Example 3, Figure 5.
Parameters taken from sheet 7.

α = .9564
β = .6627
γ = 1.8325
Log α = -.0194

(1)	(2)	(3)	(4)	(5)	(6)
X_j Variable	Log α · Col. 1	Antilog Col. 2	β · Col. 3	γ-Col. 4	Antilog Col. 5
0	0	1.0000	.6627	1.1698	14.78
7.5	-.1455	.7087	.4696	1.3629	23.06
15.0	-.2910	.5022	.3328	1.4997	31.60
22.5	-.4365	.3634	.2408	1.5917	39.06
30.0	-.5820	.2575	.1706	1.6619	45.91
37.5	-.7275	.1863	.1235	1.7090	50.22
45.0	-.8730	.1320	.0875	1.7450	55.59
60.0	-1.1640	.0677	.0449	1.7876	61.32
75.0	-1.4550	.0347	.0230	1.8095	64.49
90.0	-1.7460	.0178	.0118	1.8207	66.18
105.0	-2.0370	.0091	.0060	1.8265	67.07
120.0	-2.3280	.0047	.0031	1.8294	67.52
150.0	-2.9100	.0012	.0008	1.8317	67.88

COMPUTATION SHEET
for
Estimating Gompertz Function
(Method 2)

Description of Data: Computations for Example 4, Figure 8. Column 1 taken from Table 2.

(1)	(2)	(3)	(4)
Y_j-variable	Z_j variable $= \text{Log } Y_j$	$Z_j + 1$ variable	Col. 2 x Col. 3
44	1.64345	1.64345	2.70093
44	1.64345	1.65321	2.71697
45	1.65321	1.68124	2.77944
48	1.68124	1.71600	2.88501
52	1.71600	1.74036	2.98646
55	1.74036	1.74036	3.02885
55	1.74036	1.76343	3.06900
58	1.76343	1.78533	3.14830
61	1.78533	1.81954	3.24848
66	1.81954	1.83885	3.34586
69	1.83885	1.84510	3.39286
70	1.84510	1.86923	3.44892
74	1.86923	1.90309	3.55731
80	1.90309	1.94939	3.70986
89	1.94939	2.00000	3.89878
100	2.00000	2.04532	4.09064
111	2.04532	2.07188	4.23766
118	2.07188	2.10380	4.35882
127	2.10380	2.12385	4.46816
133	2.12385	2.12710	4.51764
134	2.12710	2.13033	4.53142
135	2.13033	2.13354	4.54514
136	2.13354	2.13672	4.55878
137	2.13672	2.13988	4.57232
138	2.13988	2.13672	4.57232

$\sum_{j=1}^{n-1} Z_j = \underline{47.60445}$

$\sum_{j=1}^{n-1} Z_j^2 = \underline{91.44561}$

$(\sum_{j=1}^{n-1} Z_j)^2 = \underline{2266.1836}$

$\sum_{j=1}^{n-1} Z_j Z_{j+1} = \underline{92.36993}$

$\sum_{j=1}^{n-1} Z_j \sum_{j=1}^{n-1} Z_{j+1} = \underline{2289.66551}$

$\sum_{j=1}^{n-1} Z_{j+1} = \underline{48.09772}$

$$B = \frac{\sum_{j=1}^{n-1} Z_j Z_{j+1} - \frac{1}{(n-1)} \sum_{j=1}^{n-1} Z_j \sum_{j=1}^{n-1} Z_{j+1}}{\sum_{j=1}^{n-1} Z_j^2 - \frac{1}{(n-1)} (\sum_{j=1}^{n-1} Z_j)^2} = \underline{.9813}$$

$$A = \frac{1}{(n-1)} \left[\sum_{j=1}^{n-1} Z_{j+1} - B \sum_{j=1}^{n-1} Z_j \right] = \underline{.0553}$$

$$\hat{\alpha} = \text{Antilog} \left(\frac{\log B}{M} \right) = \underline{.9813}$$

$$\hat{\gamma} = A/(1-B) = \underline{2.9572}$$

TABLE FOR COMPUTING VALUES OF GOMPERTZ FUNCTION

Description of Data:

Computations for Example 4, Figure 7. Parameters from Sheet 9.

α = .9813
β = 1.3138
γ = 2.9572
Log α = -.0082

(1)	(2)	(3)	(4)	(5)	(6)
X_j variable	Log α · Col. 1	Antilog Col. 2	β · Col. 3	γ - Col. 4	Antilog Col. 5
0	0	1.0000	1.3138	1.6434	44.0
2	-.0164	.9630	1.2652	1.2920	49.3
5	-.0410	.9100	1.1956	1.7616	57.8
8	-.0656	.8598	1.1296	1.8276	67.2
10	-.0820	.8280	1.0878	1.8694	74.0
12	-.0984	.7946	1.0439	1.9133	81.9
15	-.1230	.7534	.9898	1.9674	92.8
18	-.1476	.7119	.9353	2.0219	105.2
20	-.1640	.6855	.9006	2.0566	113.9
22	-.1804	.6601	.8672	2.0900	123.1
25	-.2050	.6237	.8194	2.1378	137.3
30	-.2460	.5676	.7457	2.2115	162.2
35	-.2870	.5164	.6784	2.2788	190.0
38	-.3116	.4880	.6411	2.3161	207.1

COMPUTATION SHEET
for
Estimating Spillman Function
(Method 2)

Description of Data: Computations for Example 5, Figure 10, Columns 1 and 2 taken from Table 2.

(1)	(2)	(3)
Y_j variable	$Y_j + 1$ variable	Col. 1 X Col. 2
89	100	8900
100	111	11100
111	118	13098
118	127	14986
127	133	16891
133	134	17822
134	135	18090
135	136	18360
136	137	18632
137	138	18906
138	137	18906

$\sum_{j=1}^{n-1} Y_j = \underline{1358}$

$\sum_{j=1}^{n-1} Y_j^2 = \underline{170,474}$

$(\sum_{j=1}^{n-1} Y_j)^2 = \underline{1,844,164}$

$\sum_{j=1}^{n-1} Y_j Y_{j+1} = \underline{175,691}$

$\sum_{j=1}^{n-1} Y_j \sum_{j=1}^{n-1} Y_{j+1} = \underline{1,909,348}$

$\sum_{j=1}^{n-1} Y_{j+1} = \underline{1406}$

$$B = \frac{\sum_{j=1}^{n-1} Y_j Y_{j+1} - \frac{1}{(n-1)} \sum_{j=1}^{n-1} Y_j \sum_{j=1}^{n-1} Y_{j+1}}{\sum_{j=1}^{n-1} Y_j^2 - \frac{1}{(n-1)} (\sum_{j=1}^{n-1} Y_j)^2} = \underline{.7504}$$

$$A = \frac{1}{(n-1)} \left[\sum_{j=1}^{n-1} Y_{j+1} - B \sum_{j=1}^{n-1} Y_j \right] = \underline{35.1783}$$

$\hat{\alpha} = \text{Antilog} \left(\frac{\log B}{M} \right) = \underline{.7504}$ $\hat{\gamma} = A/(1-B) = \underline{140.9387}$

TABLE FOR COMPUTING VALUES OF SPILLMAN FUNCTION

Description of Data: Computations for Example 5, Figure 9. Parameters from Sheet 11.

α = .7504
β = 51.9387
γ = 140.9387
Log α = -.12471

(1)	(2)	(3)	(4)	(5)
X variable	Log α · Col. 1	Antilog Col. 2	β · Col. 3	γ - Col. 4
0	0	1.0000	51.9387	89.0
1	-.12471	.7504	38.3408	102.6
2	-.24942	.5631	29.2467	111.7
3	-.37413	.4225	21.9441	118.9
4	-.49884	.3171	16.4698	124.5
5	-.62355	.2379	12.3562	128.6
6	-.74826	.1785	9.2711	131.7
7	-.87297	.1340	6.9598	133.9
8	-.99768	.1005	5.2198	135.7
9	-1.12239	.0754	3.9162	137.0
10	-1.24710	.0566	2.9397	138.0
11	-1.37181	.0425	2.2074	138.7
12	-1.49652	.0319	1.6568	139.3
22	-2.74362	.0018	.0935	140.8

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